A new interpretation of the Seiberg–Witten map

2003 J. Phys. A: Math. Gen. 36 L321

(http://iopscience.iop.org/0305-4470/36/21/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L321–L328 PII: S0305-4470(03)58194-X

LETTER TO THE EDITOR

A new interpretation of the Seiberg–Witten map

Subir Ghosh

Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 BT Road,
Calcutta 700 108, India

Received 23 December 2002, in final form 14 March 2003
Published 13 May 2003
Online at stacks.iop.org/JPhysA/36/L321

Abstract

In an alternative interpretation, the Seiberg–Witten map is shown to be induced by a field-dependent coordinate transformation connecting non-commutative and ordinary spacetimes. Furthermore, following our previous ideas, it has been demonstrated here that the above (field-dependent coordinate) transformation can occur naturally in the Batalin–Tyutin extended space version of the relativistic spinning particle model (in a particular gauge). There is no need to postulate the spacetime non-commutativity in an ad hoc way: it emerges from the spin degrees of freedom.

PACS numbers: 02.40.Gh, 11.10.Ef, 11.90.+t

As a natural generalization of the phase space non-commutativity (NC) in quantum mechanics, NC in *spacetime* was originally introduced by Snyder [1] as a regularization to tame the short-distance singularities, inherent in a quantum field theory (QFT). This is because NC in spacetime can introduce a lower bound in the continuity of spacetime, just as $\hbar$ does in the phase space in quantum mechanics. The advantage of NC as a regularization is that the computational scheme requires very little change from the ordinary spacetime and in some forms of NC [1] (for more recent works, see [2–4]), manifest Lorentz invariance can be maintained. However, due to the advent of renormalization techniques in QFT, Snyder's idea [1] did not gain much popularity. Also, now we know [5] that the NC prescription does not quite render a well-defined QFT, as was envisaged somewhat naively.

In more recent times, existence of non-commutativity in (open) string boundaries in the presence of a constant 2-form Neveu–Schwarz field, and the resulting non-commutative quantum field theory (NCQFT) in the branes to which the open string endpoints are attached, have rekindled interest [5] in the physics of non-commutative spacetime. Seiberg and Witten [6] have shown that the appearance of NCQFT is dependent on the choice of regularization and in fact a QFT in ordinary spacetime and an NCQFT can both describe the same underlying QFT. Concretization of this idea has lead to the celebrated Seiberg–Witten map (SWM) [6] which plays a pivotal role in our understanding of the NCQFT by directly making contact between

NCQFT and QFT in ordinary spacetime via the SWM. At least to the lowest non-trivial order in $\theta_{\mu\nu}$, the non-commutativity parameter,

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu} \tag{1}$$

the SWM can be exploited to convert an NCQFT to its counterpart living in ordinary spacetime, in which the effects of non-commutativity appear as local interaction terms, supplemented by $\theta_{\mu\nu}$. In the more popular form of NCQFT, $\theta_{\mu\nu}$ is taken to be constant. This can lead to very striking signatures in particle physics phenomenology in the form of Lorentz symmetry breakdown, new interaction vertices, etc [7].

However, as it stands, the SWM is linked exclusively to NC *gauge* theory, since the original derivation of the SWM [6] hinges on the concept of identifying gauge orbits in NC and ordinary spacetimes. In the explicit form of the SWM [6], the non-commutativity of the *spacetime* in which the NC gauge field lives, is not manifest at all since the map is a relation between the NC and ordinary gauge fields and gauge transformation parameters, all having ordinary spacetime coordinates as their arguments.

On the other hand, possibly it would have been more natural to consider first a map between NC spacetime and ordinary spacetime and subsequently to induce the SWM through the change in the spacetime argument of the gauge field from the ordinary to NC one. Precisely this type of a geometrical reformulation of the SWM is presented in this paper.

In the canonical quantization prescription, the Poisson bracket algebra is elevated to quantum commutator algebra by the replacement

$$\{A, B\} \to \frac{1}{\mathrm{i}}[\hat{A}, \hat{B}].$$

But the presence of *constraints* may demand a modification in the Poisson bracket algebra, leading to the Dirac bracket algebra [8], which are subsequently identified with the commutators

$$\{A, B\}_{\mathrm{DB}} \to \frac{1}{\mathrm{i}}[\hat{A}, \hat{B}].$$

However, complications can arise in this formalism (particularly in the case of nonlinear constraints), where the Dirac bracket algebra itself becomes operator valued. To overcome this, Batalin and Tyutin [9] have developed a systematic scheme in which all the physical variables are mapped in an extended canonical phase space, consisting of auxiliary degrees of freedom besides the physical ones, with all of them enjoying canonical free Poisson bracket algebra. In this formalism, the ambiguity of using (operator-valued) Dirac brackets as quantum commutators does not arise.

In the spinning particle model [10] the canonical $\{x_\mu, x_\nu\} = 0$ Poisson bracket changes to an operator-valued Dirac bracket,

$$\{x_\mu, x_\nu\}_{\mathrm{DB}} = -\frac{S_{\mu\nu}}{M^2} \tag{2}$$

due to the presence of constraints. In the above, the dynamical variable $S_{\mu\nu}$ represents the spin angular momentum and M is the mass of the particle. This forces us to exploit the Batalin–Tyutin prescription [9].

In a recent paper [3], we have constructed a mapping of the form

$$\{x_\mu, x_\nu\} = 0 \qquad x_\mu \to \hat{x}_\mu \qquad \{\hat{x}_\mu, \hat{x}_\nu\} = \theta_{\mu\nu} \tag{3}$$

which bridges the gap between non-commutative and ordinary spacetimes. Note that $\hat{x}_\mu$ lives in the Batalin–Tyutin [9] extended space and is of the generic form $\hat{x}_\mu = x_\mu + X_\mu$, where X_μ consists of physical and auxiliary degrees of freedom. Explicit expressions for X_μ are to be found later [3].

This spacetime map induces in a natural way the following map between non-commutative and ordinary degrees of freedom,

$$\lambda(x) \to \lambda(\hat{x}) \to \hat{\lambda}(x) \qquad A_\mu(x) \to A_\mu(\hat{x}) \to \hat{A}_\mu(x). \tag{4}$$

Here $\hat{\lambda}$ and $\hat{A}_\mu$ are the NC counterparts of λ and A_μ, the Abelian gauge transformation parameter and the gauge field respectively and $\hat{x}_\mu$ and x_μ are the NC and ordinary spacetime coordinates.

On the other hand, there also exists the SWM [6] which interpolates between non-commutative and ordinary variables,

$$\lambda(x) \to \hat{\lambda}(x) \qquad A_\mu(x) \to \hat{A}_\mu(x). \tag{5}$$

It is only logical that the above two schemes ((3)–(4) and (5)) can be related. In the present work we have precisely done that. The formulation [3] (3)–(4) being the more general one, we have explicitly demonstrated how it can be reduced to the SWM [6], in a particular gauge. This incidentally demonstrates the correctness of the procedure. The above idea was hinted in [3][1].

In this context, let us put the present work in its proper perspective. Recently a number of works [11] have appeared with the motivation of recovering the SWM in a geometric way, without invoking the gauge theory principles. However, the non-commutative feature of the spacetime plays no direct role in the above-mentioned rederivations of the SWM, with non-commutativity just being postulated in an ad hoc way. In the present work, we have shown how one can construct a non-commutative sector inside an extended phase space, in a relativistically covariant way. More importantly, we have shown explicitly how this generalized map can be reduced to the SWM under certain approximations. Interestingly, this extended space is physically significant and well studied: it is the space of the relativistic spinning particle [3, 10]. Hence it might be intuitively appealing to think that the NC spacetime is endowed with spin degrees of freedom, as compared to the ordinary configuration space, since the spin variables directly generate the NC[2]. The analogue of the gauge field is also identified inside this phase space, without any need to consider external fields. This situation is to be contrasted with the NC arising from the background magnetic field in the well-known Landau problem [5] of a charge moving in a plane in the presence of a strong, perpendicular magnetic field, or its string theory counterpart [6, 11].

We re-emphasize by mentioning that although the coordinate transformation derived here agrees with the previously obtained diffeomorphism in [11] (as it should), the framework in which it is rederived is entirely distinct from that in [11] since here we introduce a dynamical extension of the configuration space intrinsically, whereas the one in [11] requires an external gauge field. Regarding our identification of the SWM (to $O(\theta)$) as a coordinate transformation in a specific gauge in the Batalin–Tyutin extension of the spinning particle model, it should be pointed out that the choice of a particular gauge does not restrict the analogy in any way. Because of the gauge invariance of the model, other gauge choices will simply lead to gauge-equivalent theories. In fact, one can generate dual systems obeying different gauge conditions which are *not* non-commutative. Incidentally, this corroborates with the observation of Seiberg and Witten [6] that the non-commutative description of a theory is not unique. The above identification, to higher orders in θ, has not been attempted so far but the success in the $O(\theta)$

[1] The present analysis being classical, (non-)commutativity is to be interpreted in the sense of Poisson or Dirac brackets.

[2] This conjecture has been verified by us in [12], where we have explicitly constructed a non-commutative target space on a two-dimensional manifold. The NC emerges from additional target space spin fields, besides the usual spacetime coordinate degrees of freedom.

case is encouraging. Indeed, $O(\theta)$ results are relevant in themselves since most of the analysis in NC theories pertains to $O(\theta)$ computations.

The genesis of the SWM is the observation [6] that the non-commutativity in string theory depends on the choice of the regularization scheme: it appears in, e.g., point-splitting regularization whereas it does not show up in Pauli–Villars regularization. This feature, among other things, has prompted Seiberg and Witten [6] to suggest the map connecting the NC gauge fields and gauge transformation parameter to the ordinary gauge field and gauge transformation parameter. The explicit form of the SWM [6], for Abelian gauge group, to the first non-trivial order in the NC parameter $\theta_{\mu\nu}$ is the following,

$$\begin{aligned}\hat{\lambda}(x) &= \lambda(x) + \tfrac{1}{2}\theta^{\mu\nu}A_\nu(x)\partial_\mu\lambda(x)\\ \hat{A}_\mu(x) &= A_\mu(x) + \tfrac{1}{2}\theta^{\sigma\nu}A_\nu(x)F_{\sigma\mu}(x) + \tfrac{1}{2}\theta^{\sigma\nu}A_\nu(x)\partial_\sigma A_\mu(x).\end{aligned} \tag{6}$$

The above relation (6) is an $O(\theta)$ solution of the general map [6],

$$\hat{A}_\mu(A + \delta_\lambda A) = \hat{A}_\mu(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}_\mu(A) \tag{7}$$

which is based on identifying gauge orbits in NC and ordinary spacetimes.

First let us show that it is indeed possible to rederive the SWM using geometric objects. We rewrite the SWM (6) in the following way,

$$\hat{\lambda}(x) = \lambda(x) + \tfrac{1}{2}\{\delta_f[\lambda(x)] - (\lambda(x') - \lambda(x))\} = \lambda(x) + \delta_f[\lambda(x)] \tag{8}$$

$$\hat{A}_\mu(x) = A_\mu(x) + \{\delta_f[A_\mu(x)] - (A_\mu(x') - A_\mu(x))\} = A_\mu(x) + A'_\mu(x) - A_\mu(x'). \tag{9}$$

In the above we have defined

$$\begin{aligned}&x'_\mu = x_\mu - f_\mu \qquad A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}A_\nu(x) \qquad \lambda'(x') = \lambda(x)\\ &f^\mu \equiv \tfrac{1}{2}\theta^{\mu\nu}A_\nu.\end{aligned} \tag{10}$$

Here f^μ is the field-dependent spacetime translation parameter and δ_f constitutes the Lie derivative connected to f^μ,

$$\begin{aligned}\delta_f[\lambda(x)] &= \lambda'(x) - \lambda(x) = -(\lambda(x') - \lambda(x)) = f^i\partial_i\lambda\\ \delta_f[A_\mu(x)] &= A'_\mu(x) - A_\mu(x).\end{aligned} \tag{11}$$

This shows that the NC gauge parameter $(\hat{\lambda})$ and gauge field $(\hat{A}^\mu)$ are derivable from the ordinary one by making a *field-dependent* spacetime translation f^μ [13]. One can check that the NC gauge transformation of $\hat{A}_\mu(x)$ is correctly reproduced by considering

$$\hat{\delta}\hat{A}_\mu(x) = \delta(A_\mu(x) + \tfrac{1}{2}\theta^{\sigma\nu}A_\nu(x)F_{\sigma\mu}(x) + \tfrac{1}{2}\theta^{\sigma\nu}A_\nu(x)\partial_\sigma A_\mu(x)) \tag{12}$$

where $\delta A_\mu(x) = \partial_\mu\lambda(x)$ is the gauge transformation in ordinary spacetime. Hence, if expressed in the form (9), the SWM (at least to $O(\theta)$), can be derived in a geometrical way, without introducing the original identification (7) obtained from the viewpoint of a matching between NC and ordinary gauge-invariant sectors. Also note that the gauge field $A_\mu(x)$ is treated here just as an ordinary vector field, without invoking any gauge theory properties. This constitutes the first part of our result.

Returning to our starting premises, we are justified in making an identification between $\hat{x}_\mu$ in (3)–(4) and x'_μ introduced in (8)–(10), because this relation can connect NC and ordinary spacetimes. Naively, a relation of the form, $x'_\mu = x_\mu - f_\mu(x)$ cannot render the x'-space non-commutative, since the right-hand side of the equation apparently comprises commuting objects only. In our subsequent discussion we will show how this surmise can be made meaningful and return to this point at the end.

We start by considering a larger space having inherent NC. Such a space, which at the same time is physically appealing, is that of the Nambu–Goto model of relativistic spinning particle [3, 10]. Here the situation is somewhat akin to the open string boundary NC such that the role of Neveu–Schwarz field is played here by the spin degrees of freedom. The Lagrangian of the model in 2+1 dimensions [3, 10] is

$$L = \left[M^2 u^\mu u_\mu + \frac{J^2}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} + MJ\epsilon^{\mu\nu\lambda}u_\mu\sigma_{\nu\lambda}\right]^{\frac{1}{2}} \tag{13}$$

$$\begin{aligned} &u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \qquad \sigma^{\mu\nu} = \Lambda_\rho{}^\mu \frac{\mathrm{d}\Lambda^{\rho\nu}}{\mathrm{d}\tau} = -\sigma^{\nu\mu} \\ &\Lambda_\rho{}^\mu\Lambda^{\rho\nu} = \Lambda_\rho{}^\mu\Lambda^{\nu\rho} = g^{\mu\nu} \qquad g^{00} = -g^{ii} = 1. \end{aligned} \tag{14}$$

Here $(x^\mu, \Lambda^{\mu\nu})$ is a Poincaré group element and also a set of dynamical variables of the theory.

In a Hamiltonian formulation, the conjugate momenta are

$$P^\mu = \frac{\partial L}{\partial u_\mu} = L^{-1}\left[M^2 u^\mu + \frac{MJ}{2}\epsilon^{\mu\nu\lambda}\sigma_{\nu\lambda}\right] \qquad S^{\mu\nu} = \frac{\partial L}{\partial \sigma_{\mu\nu}} = \frac{L^{-1}}{2}[J^2\sigma^{\mu\nu} + MJ\epsilon^{\mu\nu\lambda}u_\lambda]. \tag{15}$$

The Poisson algebra of the above phase space degrees of freedom is

$$\{P^\mu, x^\nu\} = g^{\mu\nu} \qquad \{P^\mu, P^\nu\} = 0 \qquad \{x^\mu, x^\nu\} = 0 \qquad \{\Lambda^{0\mu}, \Lambda^{0\nu}\} = 0 \tag{16}$$

$$\{S^{\mu\nu}, S^{\lambda\sigma}\} = S^{\mu\lambda}g^{\nu\sigma} - S^{\mu\sigma}g^{\nu\lambda} + S^{\nu\sigma}g^{\mu\lambda} - S^{\nu\lambda}g^{\mu\sigma} \qquad \{\Lambda^{0\mu}, S^{\nu\sigma}\} = \Lambda^{0\nu}g^{\mu\sigma} - \Lambda^{0\sigma}g^{\mu\nu}. \tag{17}$$

The full set of constraints is

$$\Psi_1 \equiv P^\mu P_\mu - M^2 \approx 0 \qquad \Psi_2 \equiv S^{\mu\nu}S_{\mu\nu} - 2J^2 \approx 0 \tag{18}$$

$$\Theta_1^\mu \equiv S^{\mu\nu}P_\nu \qquad \Theta_2^\mu \equiv \Lambda^{0\mu} - \frac{P^\mu}{M} \qquad \mu = 0, 1, 2 \tag{19}$$

out of which Ψ_1 and Ψ_2 give the mass and spin of the particle respectively[3]. In the Dirac constraint analysis [8], these are termed as first-class constraints (FCC), having the property that they commute with *all* the constraints on the constraint surface and generate gauge transformations. The set Θ_2^μ is put by hand [3], to restrict the number of angular coordinates.

The non-commuting set of constraints $\Theta_\alpha^\mu, \alpha = 1, 2$, termed as second-class constraints (SCC) [8], modify the Poisson brackets (16) to Dirac brackets [8], defined below for any two generic variables A and B,

$$\{A, B\}_{\mathrm{DB}} = \{A, B\} - \left\{A, \Theta_\alpha^\mu\right\}\Delta_{\mu\nu}^{\alpha\beta}\left\{\Theta_\beta^\nu, B\right\} \tag{20}$$

$$\left\{\Theta_\alpha^\mu, \Theta_\beta^\nu\right\} \equiv \Delta_{\alpha\beta}^{\mu\nu} \qquad \alpha, \beta = 1, 2 \qquad \Delta_{\alpha\beta}^{\mu\nu}\Delta_{\nu\lambda}^{\beta\gamma} = \delta_\alpha^\gamma\delta_\lambda^\mu. \tag{21}$$

$\Delta_{\alpha\beta}^{\mu\nu}$ is non-vanishing even on the constraint surface. The main result relevant to us, is the following Dirac bracket [3, 10],

$$\{x_\mu, x_\nu\}_{\mathrm{DB}} = -\frac{S_{\mu\nu}}{M^2} \to \{\hat{x}_\mu, \hat{x}_\nu\} = \theta_{\mu\nu}. \tag{22}$$

[3] Note that instead of Ψ_2 as above, one can equivalently use $\Psi_2 \equiv \epsilon^{\mu\nu\lambda}S_{\mu\nu}P_\lambda - MJ$, which incidentally defines the Pauli–Lubanski scalar.

This is the non-commutativity that occurs naturally in the spinning particle model. Our aim is to express this NC coordinate $\hat{x}_\mu$ in the form $\hat{x}_\mu = x_\mu - f_\mu$, with the identification between $\theta_{\mu\nu}$ and $S_{\mu\nu}$. This is indicated in the last equality in (22). In the quantum theory, this will lead to the NC spacetime (3).

This motivates us to the Batalin–Tyutin quantization [9] of the spinning particle [3]. For a system of irreducible SCCs, in this formalism [9], the phase space is extended by introducing additional BT variables, ϕ_a^α, obeying

$$\left\{\phi_\mu^\alpha, \phi_\nu^\beta\right\} = \omega_{\mu\nu}^{\alpha\beta} = -\omega_{\nu\mu}^{\beta\alpha} \qquad \omega_{\mu\nu}^{\alpha\beta} = g_{\mu\nu}\epsilon^{\alpha\beta} \qquad \epsilon^{01} = 1 \tag{23}$$

where the last expression is a simple choice for $\omega_{\mu\nu}^{\alpha\beta}$. The SCCs Θ_α^μ are modified to $\tilde{\Theta}_\alpha^\mu$ such that they become FCC,

$$\left\{\tilde{\Theta}_\alpha^\mu(q,\phi), \tilde{\Theta}_\beta^\nu(q,\phi)\right\} = 0 \qquad \tilde{\Theta}_\alpha^\mu(q,\phi) = \Theta_\alpha^\mu(q) + \sum_{n=1}^{\infty} \tilde{\Theta}_\alpha^{\mu(n)}(q,\phi) \qquad \tilde{\Theta}^{\mu(n)} \approx O(\phi^n) \tag{24}$$

with q denoting the original degrees of freedom. Let us introduce the gauge-invariant variables $\tilde{f}(q)$ [9] corresponding to each $f(q)$, so that $\left\{\tilde{f}(q), \tilde{\Theta}_\alpha^\mu\right\} = 0$

$$\tilde{f}(q,\phi) \equiv f(\tilde{q}) = f(q) + \sum_{n=1}^{\infty} \tilde{f}(q,\phi)^{(n)} \tag{25}$$

which further satisfy [9],

$$\{q_1, q_2\}_{\mathrm{DB}} = q_3 \rightarrow \{\tilde{q}_1, \tilde{q}_2\} = \tilde{q}_3 \qquad \tilde{0} = 0. \tag{26}$$

It is now clear that our target is to obtain $\tilde{x}_\mu$ for x_μ. Explicit expressions for $\tilde{\Theta}^{\mu(n)}$ and $\tilde{f}^{(n)}$ are derived in [9].

Before we plunge into the BT analysis, the reducibility of the SCCs Θ_1^μ (i.e. $P_\mu\Theta_1^\mu = 0$) [3, 10] is to be removed [14] by introducing a canonical pair of auxiliary variables ϕ and π that satisfy $\{\phi, \pi\} = 1$ and PB commute with the rest of the physical variables. The modified SCCs that appear in the subsequent BT analysis are as shown below:

$$\Theta_1^\mu \equiv S^{\mu\nu}P_\nu + k_1 P^\mu \pi \qquad \Theta_2^\mu \equiv \left(\Lambda^{0\mu} - \frac{P^\mu}{M}\right) + k_2\left(\Lambda^{0\mu} + \frac{P^\mu}{M}\right)\phi \tag{27}$$

where k_1 and k_2 denote two arbitrary parameters. Since the computations are exhaustively done in [3] they are not repeated here. The results are the following:

$$\tilde{x}_\mu = x_\mu + [S_{\nu\mu} + 2k_1\pi g_{\nu\mu}](\phi^1)^\nu + \mathcal{R}_{1\mu\nu}(\phi^2)^\nu + \text{higher } \phi \text{ terms} \tag{28}$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = -\frac{\tilde{S}_{\mu\nu}}{M^2} \qquad \tilde{S}_{\mu\nu} = S_{\mu\nu} + \mathcal{R}_{2(\alpha)\mu\nu\lambda}\phi^{(\alpha)\lambda} + \text{higher } \phi \text{ terms} \tag{29}$$

where the expressions for $\mathcal{R}$ are straightforward to obtain [3] but are not needed in the present order of analysis. It should only be remembered that the $\mathcal{R}_1$-term in (28) is responsible for the $(\phi^\alpha)^\mu$-free term $-S_{\mu\nu}/(M^2)$ in the $\{\tilde{x}_\mu, \tilde{x}_\nu\}$ bracket in (29). Thus the problem that we had set out to solve has been addressed successfully in (28), which expresses the NC $\tilde{x}_\mu$ in terms of ordinary x_μ and other variables [3].

Now comes the crucial part of identification of the present map with the SWM [6]. This means in particular that we have to connect (28) to (10), since as we have shown before, (10) is capable of generating the SWM [6]. We exploit the freedom of choosing gauges according to our convenience, since in the BT extended space $\tilde{\Theta}_\alpha^\mu$ are FCCs. For instance, the so-called unitary gauge, $\phi_1^\mu = 0, \phi_2^\mu = 0$, trivially converts the system back to its original form before the BT extension. Let us choose the following non-trivial gauge,

$$\phi_1^\mu = \frac{M^2}{2} A^\mu(x) \qquad \phi_2^\mu = 0 \tag{30}$$

where $A^\mu(x)$ is some function of x_μ, to be identified with the gauge field. Let us also work with terms linear in $A^\mu(x)$. Identifying $\tilde{S}_{\mu\nu}/(M)^2 = \theta_{\mu\nu}$ we end up with the cherished mapping,

$$\tilde{x}_\mu = x_\mu - \tfrac{1}{2}\theta_{\mu\nu} A^\nu(x) + \text{higher } A(x) \text{ terms} \tag{31}$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\}_{\mathrm{DB}} = \theta_{\mu\nu} + \text{higher } A(x) \text{ terms.} \tag{32}$$

Note that in the above relations (31), (32), we have dropped the terms containing k_1, an arbitrary parameter [3], considering it to be very small. Also in (32) the Dirac bracket reappears since the system is gauge fixed and hence has SCCs. This constitutes the second part of our result.

Finally, two points are to be noted. Firstly, the non-commutativity present here does *not* break Lorentz invariance since there appears no constant parameter with non-trivial Lorentz index to start with. The violation will appear only in the identification of $\tilde{S}_{\mu\nu}$ with (constant) $\theta_{\mu\nu}$. Secondly, (28) truly expresses the NC spacetime $\tilde{x}_\mu$ in terms of ordinary spacetime x_μ. But x_μ becomes NC owing to the Dirac brackets induced by the particular gauge that we fixed in order to reduce our results to the SWM. Obviously, in general, there is no need to fix this particular gauge. This refers to the comment below (12).

To conclude, we have shown that it is possible to view the (Abelian $O(\theta)$) Seiberg–Witten map as a coordinate transformation involving field-dependent parameters. The idea of equivalence between gauge orbits in non-commutative and ordinary spacetimes, which was crucial in the original derivation [6], is not applied here. It has been explicitly demonstrated that a non-commutative spacetime sector can be constructed in the Batalin–Tyutin extension of the relativistic spinning particle model [3]. Finally, the above-mentioned transformation and subsequently a direct connection with the Seiberg–Witten map are also generated in this model. It emerges from the present work that non-commutative spacetime is endowed with spin degrees of freedom, as compared to the ordinary spacetime [12].

Acknowledgment

It is a pleasure to thank Professor R Jackiw for helpful correspondence.

References

[1] Snyder H S 1947 *Phys. Rev.* **71** 38
[2] Doplicher S, Fredengagen K and Roberts J E 1994 *Phys. Lett.* B **331** 39
Carlson C E, Carrone C D and Zobin N 2002 Non-commutative gauge theory without Lorentz violation *Phys. Rev.* D **66** 075001 (*Preprint* hep-th/0206035)
[3] Ghosh S 2002 *Phys. Rev.* D **66** 045031
[4] Kase H, Morita K, Okumura Y and Umezawa E 2002 Lorentz-invariant non-commutative space-time based on DFR algebra *Preprint* hep-th/0212127
[5] For a review, see Douglas M R and Nekrasov N A 2002 *Rev. Mod. Phys.* **73** 977 (*Preprint* hep-th/0106048)
Szabo R J 2001 *Preprint* hep-th/0109162
Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[6] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
[7] Calmet X *et al* 2001 *Preprint* hep-ph/0111115
Bozkaya H, Fischer P, Grosse H, Pitschmann M, Putz V, Schweda M and Wulkenhaar R 2002 Space/time noncommutative field theories and causality *Preprint* hep-th/0209253

Chaichian M, Nishijima K and Tureanu A 2002 Spin-statistics and CPT theorems in noncommutative field theory *Preprint* hep-th/0209008

Tamaki T, Harada T, Miyamoto U and Torii T 2002 Particle velocity in noncommutative space-time *Phys. Rev.* D **66** 105003 (*Preprint* gr-qc/0208002)

Carmona J M, Cortes J L, Gamboa J and Mendez F 2002 Noncommutativity in field space and Lorentz invariance violation *Preprint* hep-th/0207158

[8] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)

[9] Batalin I A and Tyutin I V 1991 *Int. J. Mod. Phys.* A **6** 3255

[10] Ghosh S 1994 *Phys. Lett.* B **338** 235

Ghosh S 2001 *J. Math. Phys.* **42** 5202

For the original works, see Hanson A J and Regge T 1974 *Ann. Phys., NY* **87** 498

See also Hanson A J, Regge T and Teitelboim C 1976 *Constrained Hamiltonian System* (Rome: Academia Nazionale dei Lincei)

[11] Cornalba L 1999 D-brane physics and noncommutative Yang–Mills theory *Preprint* hep-th/9909081

Bichl A A *et al* 2001 Noncommutative Lorentz symmetry and the origin of the Seiberg–Witten map *Preprint* hep-th/0108045

Jurco B and Schupp P 2000 *Euro. Phys. J.* C **14** 367

Jurco B, Schupp P and Wess J 2001 *Nucl. Phys.* B **584** 784

Liu H 2001 *Nucl. Phys.* B **614** 305

Jackiw R, Pi S-Y and Polychronakos A P 2002 Noncommutating gauge fields as a Lagrange fluid *Phys. Rev. Lett.* **88** 111603 (*Preprint* hep-th/0206014)

[12] Ghosh S 2002 Modelling a noncommutative two-brane *Preprint* hep-th/0212123

[13] Field-dependent transformations have appeared in, e.g., Jackiw R 1978 *Phys. Rev. Lett.* **41** 1635

Jackiw R and Pi S-Y 2001 *Preprint* hep-th/0111122

[14] Banerjee R and Barcelos-Neto J 1998 *Ann. Phys., NY* **265** 134